

## A NEW PERSPECTIVE ON FEASIBILITY DETERMINATION

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### ABSTRACT

We consider the problem of feasibility determination in a stochastic setting. In particular, we wish to determine whether a system belongs to a given set  $\Gamma$  based on a performance measure estimated through Monte Carlo simulation. Our contribution is two-fold: (i) we characterize fractional allocations that are asymptotically optimal; and (ii) we provide an easily implementable algorithm, rooted in stochastic approximation theory, that results in sampling allocations that provably achieve in the limit the same performance as the optimal allocations. The finite-time behavior of the algorithm is also illustrated on two small examples.

### 1 BACKGROUND

We consider the problem of feasibility determination in a stochastic setting. In particular, we wish to determine whether a system belongs to a given set  $\Gamma$  based on a performance measure estimated through Monte Carlo simulation. While it is an interesting problem in its own right, feasibility determination has recently attracted much attention from researchers in ranking and selection (R&S) whose objective is to select the best system in the presence of a stochastic constraint.

Ranking and selection (R&S) techniques are statistical methods developed to select the best system, or a subset of systems from among a set of alternative system designs. R&S via simulation is particularly appealing as it combines modeling flexibility of simulation with the efficiency of statistical techniques for effective decision making. Furthermore, it is relatively straightforward to satisfy the underlying technical assumptions of these techniques in simulation experiments, which also allow for multi-stage sampling as required by some R&S methods.

Due to randomness in output data, comparing a number of simulated systems requires care. Suppose we conduct  $n$  simulation replications for each of  $r$  designs. Therefore, we need  $rn$  simulation replications. Simulation results become

more precise as  $n$  increases. If the precision requirement is high ( $n$  is not small), and if the total number of designs in a decision problem is large ( $r$  is large), then  $rn$  can be very large, which may easily render the total simulation cost prohibitively high and preclude the feasibility of using simulation for R&S problems. The effective employment of the simulation budget in the course of obtaining a good decision is therefore crucial.

The rich literature on R&S has been developing along two principal axes. The optimal computing budget allocation (OCBA) approach has its roots in [Dudewicz and Dalal \(1975\)](#) who developed a two-stage procedure for selecting the best design or a design that is very close to the best system. [Rinott \(1978\)](#) presents an alternative way to compute the required number of simulation replications in the second stage. This idea has been ultimately extended to fully sequential algorithms. [Paulson \(1964\)](#), [Gupta and Panchapakesan \(1979\)](#), [Nelson and Matejcik \(1995\)](#), [Matejcik and Nelson \(1995\)](#), [Hsu \(1996\)](#), and [Kim and Nelson \(2006\)](#) present methods based on the classical statistical model adopting a frequentist view. On the other hand, the value of information (VIP) approach, exemplified by [Berger \(1985\)](#), [Bernardo and Smith \(1994\)](#), [Gupta and Miescke \(1996\)](#), [Chick and Inoue \(2001\)](#), and [Chen et al. \(2000\)](#) uses a Bayesian framework for managing the trade-off between the consequences of an immediate decision and the cost of additional sampling.

The overwhelming majority of the R&S research focuses on a single unconstrained performance measure ([Kim and Nelson 2006](#)). Most applications, however, either have multiple performance measures ([Butler, Morrice, and Mullarkey 2001](#)) or face some constraints on the primary performance measure ([Andradottir, Goldsman, and Kim 2005](#)). In a manufacturing setting, for example, one might be simultaneously interested in selecting the scheduling policy that maximizes throughput, with a physical limit on the work in process or a service constraint on lead times. While some of these measures might be correlated, others might simply conflict with one another. The literature dealing with

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multiple performance measures or stochastic constraints has been sparse.

In the case with multiple performance measures, only a partial ordering may be possible, i.e., these populations may not be ordered in terms of a vector-valued parameter. In such cases, one typically defines a real-valued function of the parameters and use this function to rank the populations. [Gupta et al. \(1973\)](#) propose several such scalar functions. [Andijani \(1998\)](#) uses the analytic hierarchical process (AHP) and [Butler et al. \(2001\)](#) use the multiple attribute utility theory (MAUT) to construct a scalar function by assigning weights to the performance measures. [Kim and Lin \(2001\)](#) use a *max-min* approach whereby they maximize the poorest performing criterion. [Lee et al. \(2006\)](#) and [Lee et al. \(2007\)](#) seek to identify a Pareto set of non-dominated designs.

R&S in the presence of stochastic constraints has recently started receiving some attention. In a setting where R&S is based on a primary performance measure subject to the feasibility of a (possibly correlated) secondary performance measure, [Andradottir et al. \(2005\)](#) propose a two-phase approach : phase I identifies feasible systems while phase II selects the best among them. With the objective of accelerating the first phase, [Batur and Kim \(2005\)](#) have recently introduced procedures for finding a set of feasible or near-feasible systems. The present article also is concerned with efficient feasibility determination. Unlike the previous work that is based on the indifference zone perspective, we use large deviations theory to minimize the expected number of incorrect determinations. We characterize the optimal employment of the simulation budget to sampling from each system, and present a stochastic approximation algorithm that yields a budget allocation that provably converges to the optimal one. It is worth pointing out that we are not the first ones using the large deviations framework for R&S. [Glynn and Juneja \(2004\)](#) have used this framework not only to determine an optimal budget allocation strategy, but to also demonstrate the deteriorating effect of violating the normality assumptions, which are very commonly used in the R&S literature. Our notation closely follows theirs. Like them, we characterize the optimal allocation of the computational budget. However, we also provide an algorithm to achieve the optimal sampling allocations.

The remainder of the paper is organized as follows. Section 2 introduces the necessary notation along with some preliminary ideas from the large deviations theory. Section 3 introduces our algorithm for feasibility determination. A numerical illustration is presented in Section 4. Section 5 concludes the paper.

## 2 OPTIMAL ASYMPTOTIC ALLOCATION

We consider  $r$  systems, each with unknown performance measure  $\mu_1, \dots, \mu_r \in R$ . Given a set  $\Gamma = [\gamma, \infty)$ , we wish

to employ Monte Carlo simulation to determine for each system  $i$  whether  $\mu_i \in \Gamma$  or not. We assume the simulationist can obtain i.i.d. replicates of the random variable  $X_i$  to form unbiased estimators  $\bar{X}_i(n_i) = n_i^{-1} \sum_{k=1}^{n_i} X_{i,k}$  of  $\mu_i$ , for  $i = 1, \dots, r$ .

The simulation budget  $n$  is allocated in order to minimize the expected number of incorrect determinations. If, without loss of generality, we assume that  $\mu_1, \dots, \mu_a \in \Gamma$  and  $\mu_{a+1}, \dots, \mu_r \notin \Gamma$ , then the feasibility determination problem is to

$$\min_{p_1, \dots, p_r \in \mathcal{M}} g_n(p_1, \dots, p_r)$$

where

$$g_n(p_1, \dots, p_r) = \sum_{i=1}^a P(\bar{X}_i(p_i n) \notin \Gamma) + \sum_{i=a+1}^r P(\bar{X}_i(p_i n) \in \Gamma),$$

and  $\mathcal{M} = \{p_i \geq 0 : \sum_i p_i \leq 1\}$ . The  $p_i$ 's represent the fraction of the simulationist's budget that is allocated to sampling from each system, where for simplicity we assume that each system has the same per-sample cost.

Our contribution is two-fold: (i) we characterize fractional allocations  $p_1^*, \dots, p_r^*$  that are optimal (in log scale) as  $n \rightarrow \infty$ ; and (ii), we provide an easily implementable algorithm, rooted in stochastic approximation theory, that results in sampling allocations that provably achieve the same performance (in log scale) as the optimal allocations in the limit as  $n \rightarrow \infty$ .

Let  $I_i(\cdot)$  be the large deviations rate function

$$I_i(x) = \sup_{\theta \in R} \{\theta x - \log M_i(\theta)\}, \quad (1)$$

where

$$M_i(\theta) = E \exp(\theta X_i)$$

is the moment generating function, for each system  $i = 1, \dots, r$ . Our main assumptions are that:

- A1. The performance measures do not lie exactly at the boundary:  $\mu_i \neq \gamma$  for all  $i = 1, \dots, r$ .
- A2. For all systems,  $M_i(\theta) < \infty$  for  $\theta \in R$ .

Assumption A1 ensures that the rate functions  $I_i(\cdot)$  evaluated at the boundary point  $\gamma$  are positive, so that no system requires all the simulation budget. Assumption A2 is standard in situations where the underlying distributions are light tailed, and implies that for each system there exists  $\theta_i^* \in R$  (unique) such that  $I_i(\gamma) = \theta_i^* \gamma - M_i(\theta_i^*)$ . It can be shown that  $\theta_i^* < 0$  for  $i \in \{1, \dots, a\}$  and  $\theta_i^* > 0$  for  $i \in \{a+1, \dots, r\}$ .

We are now ready to state the main results of this section.

**Proposition 1** Suppose Assumption A2 holds. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log g_n(p_1, \dots, p_r) = -\min_i p_i I_i(\gamma). \quad (2)$$

**Proof:** The proof consists of two parts. For  $p_i > 0$  and  $\theta$  scalar

$$\frac{1}{n} \log E(\exp(n\theta \bar{X}_i(p_i n))) = p_i \log M_i(\theta/p_i).$$

Moreover,

$$\begin{aligned} \sup_{\theta \in \mathbb{R}} \{\gamma\theta - p_i \log M_i(\theta/p_i)\} &= p_i \sup_{\theta \in \mathbb{R}} \{\gamma\theta/p_i - \log M_i(\theta/p_i)\} \\ &= p_i I_i(\gamma). \end{aligned}$$

Hence, the Gärtner-Ellis Theorem implies that  $1/n \log P(\bar{X}_i(np_i) \notin \Gamma) \rightarrow -p_i I_i(\gamma)$  for  $i = 1, \dots, a$ , and  $1/n \log P(\bar{X}_i(np_i) \in \Gamma) \rightarrow -p_i I_i(\gamma)$  for  $i = a+1, \dots, r$ , as  $n \rightarrow \infty$ .

In the second part we obtain lower and upper bounds for  $(1/n) \log g_n(p_1, \dots, p_r)$ . Suppose that  $p_{i^*} I_{i^*} = \min_i p_i I_i(\gamma)$ . Then, since  $g_n(p_1, \dots, p_r) \geq P(\bar{X}_{i^*}(np_{i^*}) \notin \Gamma)$  if  $i^* \in \{1, \dots, a\}$  and  $g_n(p_1, \dots, p_r) \geq P(\bar{X}_{i^*}(np_{i^*}) \in \Gamma)$  if  $i^* \in \{a+1, \dots, r\}$ , we obtain the lower bound  $\liminf_{n \rightarrow \infty} (1/n) \log g_n(p_1, \dots, p_r) \geq -p_{i^*} I_{i^*}(\gamma)$ . For the upper bound,

$$\begin{aligned} \frac{1}{n} \log g_n(p_1, \dots, p_r) &\leq \frac{1}{n} \max\{\log(rP(\bar{X}_1(np_1) \notin \Gamma)), \dots, \\ &\quad \log(rP(\bar{X}_a(np_a) \notin \Gamma)), \log(rP(\bar{X}_1(np_{a+1}) \in \Gamma)), \dots, \\ &\quad \log(rP(\bar{X}_1(np_r) \in \Gamma))\}. \end{aligned}$$

Taking limits,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log g_n(p_1, \dots, p_r) \leq -p_{i^*} I_{i^*}(\gamma).$$

⊗

The best achievable exponential decay rate for  $g_n(\cdot)$  is described next.

**Proposition 2** Suppose assumptions A1 and A2 hold. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log g_n(p_1^*, \dots, p_r^*) = -\frac{1}{\sum_{j=1}^r I_j^{-1}(\gamma)}$$

where

$$p_i^* = \frac{I_i^{-1}(\gamma)}{\sum_{j=1}^r I_j^{-1}(\gamma)}. \quad (3)$$

For any other  $(p_1, \dots, p_r) \in \mathcal{M}$  we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log g_n(p_1, \dots, p_r) \geq -\frac{1}{\sum_{j=1}^r I_j^{-1}(\gamma)}.$$

**Proof:** The first statement follows immediately by evaluating Eq. (2) with  $p_i^*$ . For the second statement we need to show that  $p_1^*, \dots, p_r^*$  maximize the exponential decay rate of  $g_n(\cdot)$ , given in Eq.(2). Hence we consider the problem  $\max_{p \in \mathcal{M}} \min_i p_i I_i(\gamma)$ , which is equivalent to the linear problem

$$\begin{aligned} &\max \eta \\ \text{st} \quad &\eta - p_i I_i(\gamma) \leq 0 \text{ for } i = 1, \dots, r \\ &\sum_{j=1}^r p_j \leq 1 \\ &p_i \geq 0 \text{ for } i = 1, \dots, r \end{aligned}$$

The dual of this problem is

$$\begin{aligned} &\min \pi \\ \text{st} \quad &\pi - q_i I_i(\gamma) \geq 0 \text{ for } i = 1, \dots, r \\ &\sum_{j=1}^r q_j \geq 1 \\ &\pi, q_i \geq 0 \text{ for } i = 1, \dots, r \end{aligned}$$

The variables  $p_i = q_i = I_i^{-1}(\gamma) / \sum_{j=1}^r I_j^{-1}(\gamma)$ ,  $\eta = \pi = 1 / \sum_{j=1}^r I_j^{-1}(\gamma)$  are primal and dual feasible and  $\eta = \pi$ , so that  $p_i^*$  is primal optimal by weak duality. ⊗

We view  $p_i^*$  as the optimal allocation scheme, meaning that no other allocation achieves a higher exponential decay rate for  $g_n(\cdot)$  as the sampling budget goes to infinity.

**Example 1** Suppose for every system we have  $X_i \sim N(\mu_i, \sigma_i^2)$ , with  $\sigma_i^2 > 0$ . For each system  $i$  (see Exercise 2.2.23 of [Dembo and Zeitouni 1998](#)),

$$I_i(\gamma) = \frac{1}{2} \left( \frac{\mu_i - \gamma}{\sigma_i} \right)^2.$$

Hence Proposition 2, specialized to the Normal case, becomes

$$p_i^* = \frac{\sigma_i^2 / (\mu_i - \gamma)^2}{\sum_{j=1}^r \sigma_j^2 / (\mu_j - \gamma)^2}. \quad (4)$$

In other words, when the underlying distributions are normal, each system is sampled in proportion to the square of the distance of its scaled performance measure to the boundary point  $\gamma$ .

**Example 2** Assume  $X_i \sim \text{Bernoulli}(\mu_i)$ ,  $0 < \mu_i < 1$ , for  $i = 1, \dots, r$ , and let  $0 < \gamma < 1$ . The rate functions are

(Exercise 2.2.23 of [Dembo and Zeitouni 1998](#)),

$$I_i(\gamma) = \gamma \log \left( \frac{\gamma}{\mu_i} \right) + (1 - \gamma) \log \left( \frac{1 - \gamma}{1 - \mu_i} \right),$$

so that Proposition 2 results in

$$p_i^* = \frac{\left[ \gamma \log \left( \frac{\gamma}{\mu_i} \right) + (1 - \gamma) \log \left( \frac{1 - \gamma}{1 - \mu_i} \right) \right]^{-1}}{\sum_{j=1}^r \left[ \gamma \log \left( \frac{\gamma}{\mu_j} \right) + (1 - \gamma) \log \left( \frac{1 - \gamma}{1 - \mu_j} \right) \right]^{-1}}. \quad (5)$$

While Proposition 2 characterizes the optimal sampling allocation, the rate functions  $I_i(\gamma)$  are generally unknown and need to be estimated. However, estimating each rate function involves solving a root finding problem, as can be seen in Eq. (1) by solving for the derivative with respect to  $\theta$  equal to zero. This suggests that the computational cost of sequentially estimating the rate functions dominates all other costs when running the simulation, and therefore is not a promising course of action. In the next section we present a stochastic approximation algorithm that overcomes these issues, and leads to fractional allocations that converge almost surely to the optimal  $p_i^*$  allocations.

### 3 STOCHASTIC APPROXIMATION ALGORITHM

Initially we warm-up each system with  $n_0 > 0$  replicates and compute the sample averages  $\tilde{X}_{i,0} = n_0^{-1} \sum_{k=1}^{n_0} X_{i,k}$ . Let

$$\begin{aligned} I_{i,0} &= \sup_{\theta \in R} \left\{ \theta \gamma - \log \left( \frac{1}{n_0} \sum_{k=1}^{n_0} \exp(\theta X_{i,k}) \right) \right\} \\ &= \theta_{i,0} \gamma - \log(M_{i,0}), \end{aligned}$$

where  $\theta_{i,0}$  satisfies the root problem

$$\gamma = \frac{\sum_{k=1}^{n_0} X_{i,k} \exp(\theta_{i,0} X_{i,k})}{\sum_{k=1}^{n_0} \exp(\theta_{i,0} X_{i,k})},$$

and the sample moment generating functions  $M_{i,0}$  evaluated at  $\theta_{i,0}$  are given by

$$M_{i,0} = \frac{1}{n_0} \sum_{k=1}^{n_0} \exp(\theta_{i,0} X_{i,k}).$$

In order for the algorithm to get properly initialized, if  $I_{i,0} \leq 0$  we set  $I_{i,0} = \varepsilon$ , for  $\varepsilon$  a small positive constant.

Finally, we initialize the sample sizes,  $\lambda_{i,0} = n_0$  for  $i = 1, \dots, r$ .

#### Algorithm.

1. Initialize  $n = 0$ .

2. Generate a replicate  $\alpha$  from the p.m.f.  $I_{i,n}^{-1} / \sum_{j=1}^r I_{j,n}^{-1}$ , for  $i = 1, \dots, r$ .
3. Update sample sizes:  $\lambda_{\alpha,n+1} = \lambda_{\alpha,n} + 1$ , and  $\lambda_{i,n+1} = \lambda_{i,n}$  for  $i \neq \alpha$ .
4. Generate a replicate from system  $\alpha$ , (say)  $X_{\alpha,\lambda_\alpha}$ .
5. Update  $\tilde{X}_{\alpha,n+1}$ ,  $M_{\alpha,n+1}$ ,  $\theta_{\alpha,n+1}$ , and  $I_{\alpha,n+1}$ :

$$\tilde{X}_{\alpha,n+1} = \tilde{X}_{\alpha,n} + \frac{1}{\lambda_{\alpha,n+1}} (X_{\alpha,\lambda_\alpha} - \tilde{X}_{\alpha,n}),$$

$$M_{\alpha,n+1} = M_{\alpha,n} + \frac{1}{\lambda_{\alpha,n+1}} (\exp(\theta_{\alpha,n} X_{\alpha,\lambda_\alpha}) - M_{\alpha,n}),$$

$$\begin{aligned} \theta_{\alpha,n+1} &= \theta_{\alpha,n} \\ &\quad - \frac{1}{\lambda_{\alpha,n+1}} (X_{\alpha,\lambda_\alpha} - \gamma) \exp(\theta_{\alpha,n} (X_{\alpha,\lambda_\alpha} - \gamma)), \end{aligned}$$

and

$$I_{\alpha,n+1} = \theta_{\alpha,n+1} \gamma - \log(M_{\alpha,n+1}),$$

if  $\theta_{\alpha,n+1} \gamma - \log(M_{\alpha,n+1}) > 0$ , and  $I_{\alpha,n+1} = \min_j I_{j,n}$  otherwise. (This ensures that  $I_{\alpha,n+1} > 0$  always.) For  $i \neq \alpha$ , set  $\tilde{X}_{i,n+1} = \tilde{X}_{i,n}$ ,  $M_{i,n+1} = M_{i,n}$ ,  $\theta_{i,n+1} = \theta_{i,n}$ , and  $I_{i,n+1} = I_{i,n}$ .

6. Increase  $n \leftarrow n + 1$  and go back to 2.

To ensure that each system is sampled infinitely often, let  $(v_n)$  be an increasing sequence such that  $v_n \rightarrow \infty$  and  $n^{-1} \sum_{k=1}^n J(v_k \leq n) \rightarrow 0$ , where  $J(\cdot)$  is the indicator function. We sample from all  $r$  systems at stages  $v_1, v_2, \dots$ , and update the parameters according to steps 3 and 5 of the algorithm.

This algorithm provides fractional allocations that converge a.s. to the optimal allocations, as the next theorem demonstrates.

**Theorem 1** Under assumptions A1 and A2,

$$\frac{\lambda_{i,n}}{n} \rightarrow p_i^*,$$

almost surely as  $n \rightarrow \infty$ .

**Proof outline:** Suppose momentarily that  $\theta_{i,n} \rightarrow \theta_i^*$  a.s. Let  $p_{i,n} = \lambda_{i,n}/n$ , so that step 3 of the algorithm can be expressed as  $p_{i,n+1} = p_{i,n} + (J(\alpha_n = i) - p_{i,n})/(n+1)$ , where  $\alpha_n$  is the  $n$ th replicate of  $\alpha$  generated in step 2 of the algorithm, and  $J(\cdot)$  is the indicator function. The recursion for  $p_{i,n+1}$  can be re-written as

$$\begin{aligned} p_{i,n+1} &= p_{i,n} + \frac{1}{n+1} (p_i^* - p_{i,n}) + \frac{1}{n+1} (J(\alpha_n = i) - q_{i,n}) \\ &\quad + \frac{1}{n+1} (q_{i,n} - p_i^*), \quad (6) \end{aligned}$$

where  $q_{i,n} = I_{i,n}^{-1} / \sum_{j=1}^r I_{j,n}^{-1}$ . Since  $E[J(\alpha_n = i) | I_{j,0}, \alpha_k, X_{\alpha_k, \lambda_{\alpha_k}}, j = 1, \dots, r, k = 0, \dots, n] = q_{i,n}$ , the sequence  $\{J(\alpha_n = i) - q_{i,n}\}$  is a martingale difference with respect to the sequence of  $\sigma$ -algebras generated by  $\{I_{j,0}, \theta_{j,0}, \alpha_k, X_{\alpha_k, \lambda_{\alpha_k}}, j = 1, \dots, r, k = 0, \dots, n\}$ . Additionally,

$$\sum_k \frac{|q_{i,k} - p_i^*|}{k+1} \leq \sum_k \frac{J(|q_{i,k} - p_i^*| > 1/(1+k))}{k+1} + \sum_k \frac{1}{(k+1)^2}, \quad (7)$$

almost surely. The set  $\{\sum_k J(|q_{i,k} - p_i^*| > 1/(1+k))/(k+1) = \infty\} \subseteq \cup_{\ell=0}^{\infty} \{\sum_k J(|q_{i,k} - p_i^*| > 1/(1+\ell))/(k+1) = \infty\}$ , and  $P(\sum_k J(|q_{i,k} - p_i^*| > 1/(1+\ell))/(k+1) = \infty) = 0$  for each  $\ell$  since  $\theta_{i,n} \rightarrow \theta_i^*$  a.s. implies  $q_{i,n} \rightarrow p_i^*$  a.s. Therefore  $P(\sum_k J(|q_{i,k} - p_i^*| > 1/(1+k))/(k+1) = \infty) = 0$ , and we conclude from Eq. (7), that  $\sum_k |q_{i,k} - p_i^*|/(k+1) < \infty$  a.s.

Hence, all the assumptions of Theorem 5.2.1 of [Kushner and Yin \(2003\)](#) are satisfied. Moreover, the ordinary differential equations  $p'_i = p_i^* - p_i$ ,  $i = 1, \dots, r$ , have a unique globally asymptotically stable point,  $p_i^*$ , so that Theorem 5.2.1 of [Kushner and Yin \(2003\)](#) results in  $p_{i,n} \rightarrow p_i^*$  a.s. (The special instances of  $n = v_k, k = 1, 2, \dots$ , can be handled by introducing an extra term in Eq. (6), and do not affect the asymptotic behavior of  $p_{i,n}$ .)

Regarding the asymptotic behavior of  $\theta_{i,n}$ , let  $\bar{g}_i(\theta_{i,n}) = E[(X_i - \gamma) \exp(\theta_{i,n}(X_i - \gamma))]$  and re-write the recursion for  $\theta_{i,n}$  as

$$\theta_{\alpha,n+1} = \theta_{\alpha,n} + \frac{1}{\lambda_{\alpha,n+1}} \times \left( \bar{g}_\alpha(\theta_{\alpha,n}) - (X_{\alpha,\lambda_\alpha} - \gamma) \exp(\theta_{\alpha,n}(X_{\alpha,\lambda_\alpha} - \gamma)) \right) - \frac{\bar{g}_\alpha(\theta_{\alpha,n})}{\lambda_{\alpha,n+1}}.$$

The sequence  $\{\bar{g}_\alpha(\theta_{\alpha,n}) - (X_{\alpha,\lambda_\alpha} - \gamma) \exp(\theta_{\alpha,n}(X_{\alpha,\lambda_\alpha} - \gamma))\}$  is a martingale difference with respect to the sequence of  $\sigma$ -algebras generated by  $\{I_{j,0}, \theta_{j,0}, \alpha_k, X_{\alpha_k, \lambda_{\alpha_k}}, j = 1, \dots, r, k = 0, \dots, n\}$ . Since  $\lambda_{i,n} \rightarrow \infty$  a.s., Theorem 5.2.1 of [Kushner and Yin \(2003\)](#) applies, meaning that  $\theta_{i,n}$  converges a.s. to the unique globally asymptotically stable point of the ODE

$$\theta'_i = -\bar{g}_i(\theta_i),$$

which is given by  $\theta_i^*$ , for  $i = 1, \dots, r$ .  $\otimes$

**Remark.**  $p_i^* I_i(\gamma) = p_j^* I_j(\gamma)$  suggests that step 2 of the algorithm could be replaced by  $\alpha = \arg \min_i \{\lambda_{i,n} I_{i,n}\}$ . An argument similar to the one employed in the proof of Theorem 1 shows that this approach also results in asymptotically optimal fractional allocations.

The next result shows that the expected number of incorrectly determined systems produced by the stochastic approximation algorithm approaches 0 at the best possible rate.

**Theorem 2** Suppose that assumptions A1 and A2 hold. Then,

$$\frac{1}{n} \log \left( \sum_{i=1}^a P(\tilde{X}_{i,n} \notin \Gamma) + \sum_{i=a+1}^r P(\tilde{X}_{i,n} \in \Gamma) \right) \rightarrow -\frac{1}{\sum_{j=1}^r I_j^{-1}(\gamma)},$$

as  $n \rightarrow \infty$ .

**Proof outline:** The proof resembles the proof of Proposition 1. First of all, the random variables  $X_{i,1}, \dots, X_{i,\lambda_{i,n}}$  are i.i.d. conditional on  $\lambda_{i,n}$ , so that

$$\frac{1}{n} \log (E \exp(n\theta \tilde{X}_{i,n})) = \frac{1}{n} \log E \left[ M_i \left( \theta \frac{n}{\lambda_{i,n}} \right)^{\lambda_{i,n}} \right],$$

where the expectation is with respect to the probability measure of the random variable  $\lambda_{i,n}$ . Assumptions A1 and A2, Theorem 1, and the continuity of  $M_i(\cdot)$  lead to

$$M_i \left( \theta \frac{n}{\lambda_{i,n}} \right)^{\lambda_{i,n}/n} \rightarrow M_i \left( \frac{\theta}{p_i^*} \right)^{p_i^*} \quad (8)$$

a.s. By Jensen's inequality

$$\left( E \left[ M_i \left( \theta \frac{n}{\lambda_{i,n}} \right)^{\lambda_{i,n}/n} \right] \right)^n \leq E \left[ M_i \left( \theta \frac{n}{\lambda_{i,n}} \right)^{\lambda_{i,n}} \right],$$

so that

$$p_i^* \log M_i \left( \frac{\theta}{p_i^*} \right) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log (E \exp(n\theta \tilde{X}_{i,n})). \quad (9)$$

By Eq. (8), given  $\varepsilon > 0$ ,

$$M_i \left( \theta \frac{n}{\lambda_{i,n}} \right)^{\lambda_{i,n}/n} \leq M_i \left( \frac{\theta}{p_i^*} \right)^{p_i^* + \varepsilon}$$

a.s. for all  $n$  sufficiently large, which results in

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log (E \exp(n\theta \tilde{X}_{i,n})) \leq (p_i^* + \varepsilon) \log M_i \left( \frac{\theta}{p_i^*} \right).$$

Sending  $\varepsilon \rightarrow 0$ , together with Eq. (9), yields

$$\frac{1}{n} \log (E \exp(n\theta \tilde{X}_{i,n})) \rightarrow p_i^* \log M_i \left( \frac{\theta}{p_i^*} \right).$$

By the Gärtner-Ellis theorem, we conclude that  $1/n \log P(\tilde{X}_{i,n} \notin \Gamma) \rightarrow -p_i^* I_i(\gamma)$  for  $i = 1, \dots, a$ , and  $1/n \log P(\tilde{X}_{i,n} \in \Gamma) \rightarrow -p_i^* I_i(\gamma)$  for  $i = a+1, \dots, r$ , as  $n \rightarrow \infty$ .

The rest of the proof is similar to the second part of the proof of Proposition 1, and is thus omitted.  $\otimes$

#### 4 NUMERICAL EXAMPLES

We now illustrate our stochastic approximation algorithm through the two examples introduced in Section 2, where the performance metric of interest is distributed according to Normal and Bernoulli distributions, respectively. In addition to providing us with an opportunity to examine the finite-time performance of our algorithm, these two examples also illustrate the impact of deviations from the assumption of normality, which is quite common in the R&S literature. All the results are the average of 100 independent mega-replications. The results, which reflect the allocation of the computing budget as a function of the number of iterations of the algorithm, lead to two key observations: the efficacy of sequential algorithms and the importance of adequate initialization of such algorithms. The efficacy of these algorithms is particularly valuable in critical settings, i.e., those settings where the system performance is close to the boundary of the feasible region.

**Example 3** Suppose we have five systems with  $X_i \sim N(\mu_i, \sigma_i^2)$ , with  $\mu_i = [9.51, 9.45, 9.40, 9.55, 9.60]$  and  $\sigma_i^2 = 1$ . Let the boundary point  $\gamma = 9.50$ . As a starting point, we will use Eq. (4), and sample from each system in proportion to the square of the distance of its scaled performance measure to  $\gamma$ . In this case, the optimal budget allocation for the five systems is given by  $p_i^* = [0.9091, 0.0364, 0.0091, 0.0364, 0.0091]$ . Figure 1 shows that convergence is rather rapid, that is, within 100000 iterations, the optimal allocation of the computing budget is achieved along with the correct classification of all the systems. Note that the algorithm has been initialized with only 100 samples from each system.

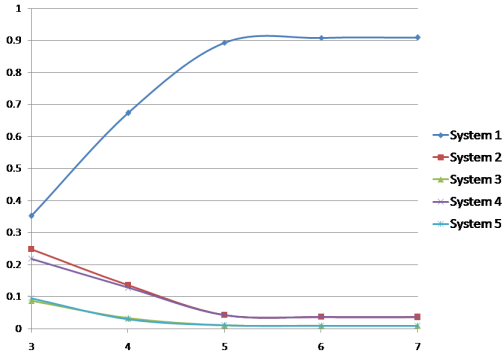


Figure 1: Convergence of the budget allocation as a function of log(iterations) for the Normal case with the exact rate function

Given the fact that the rate functions are generally unknown and need to be estimated, we next deploy our stochastic approximation algorithm to classify five systems with  $X_i \sim N(\mu_i, \sigma_i^2)$ , with  $\mu_i = [9.20, 8.50, 9.00, 9.80, 10.01]$  and  $\sigma_i^2 = 1$ . Let the boundary point  $\gamma = 9.50$ . In this case, the optimal budget allocation for the five systems is given by  $p_i^* = [0.3577, 0.0322, 0.1288, 0.3577, 0.1238]$ . Figure 2 shows that convergence has slowed down due to the effort required in estimating the rate functions; that is, it now takes over 1000000 iterations for the stochastic approximation algorithm to achieve the optimal allocation of the computing budget along with the correct classification of all the systems. Note that the algorithm has been initialized with 50000 samples from each system.

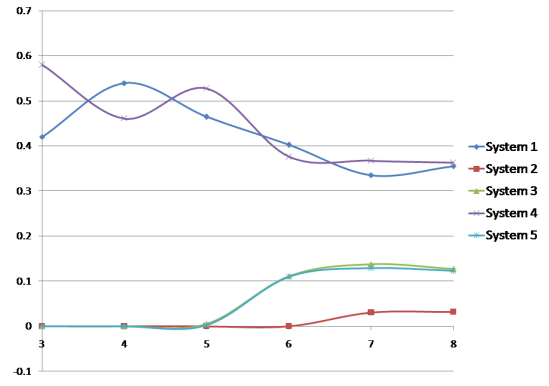


Figure 2: Convergence of the budget allocation as a function of log(iterations) for the Normal case with the estimated rate function

**Example 4** Assume  $X_i \sim \text{Bernoulli}(\mu_i)$ , with  $\mu_i = [0.92, 0.85, 0.90, 0.98, 0.88]$ . Let the boundary point  $\gamma = 0.95$ . Based on Eq. (5), the optimal budget allocation for the five systems is given by  $p_i^* = [0.4492, 0.0618, 0.1878, 0.1927, 0.1084]$ . Figure 3 shows that, within 10000 iterations, our stochastic approximation algorithm achieves the optimal allocation of the computing budget along with the correct classification of all the systems. One must recall, however, that appropriate initialization of the algorithm is crucial. The above performance is obtained after collecting 50000 observations to initialize the algorithm.

#### 5 CONCLUDING REMARKS

We address the problem of feasibility determination in a stochastic setting. More specifically, we wish to determine whether a system belongs to a given set  $\Gamma$  based on a performance measure estimated through Monte Carlo simulation. Our contribution is two-fold: (i) we characterize fractional allocations that are asymptotically optimal; and (ii) we provide an easily implementable algorithm, rooted in stochastic approximation theory, that results in sampling allocations

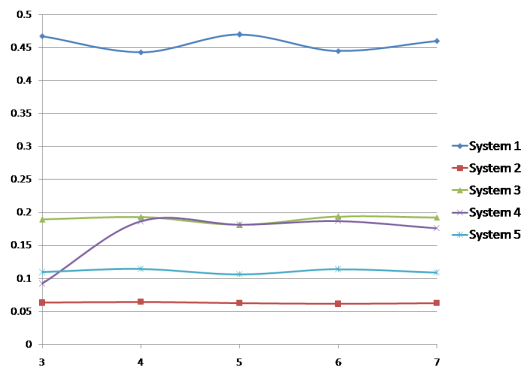


Figure 3: Convergence of the budget allocation as a function of  $\log(\text{iterations})$  for the Bernoulli case with the estimated rate function

that provably achieve in the limit the same performance as the optimal allocations. The finite-time behavior of the algorithm, illustrated on two small examples, is quite promising. Current research is aimed at generalizing the approach to settings where the feasible region is  $d$ -dimensional with  $d > 1$ .

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